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Nonclassical symmetry reductions of the three-dimensional incompressible Navier–Stokes equations

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Abstract. The nonclassical reduction method as pioneered by Bluman and Cole (*J. Meth. Mech.* **18** 1025–42) is used to examine symmetries of the full three-dimensional, unsteady, incompressible Navier–Stokes equations of fluid mechanics. The procedure, when applied to a system of partial differential equations, yields reduced sets of equations with one fewer independent variables. We find eight possibilities for reducing the Navier–Stokes equations in the three spatial and one temporal dimensions to sets of partial differential equations in three independent variables. Some of these reductions are derivable using the Lie-group method of classical symmetries but the remainder are genuinely nonclassical. Further investigations of one of our eight forms shows how it is possible to derive novel exact solutions of the Navier–Stokes equations by the nonclassical method.

1. Introduction

The overwhelming majority of important governing equations in physics take the form of nonlinear partial differential equations, commonly systems of equations, and so are frequently virtually impossible to solve explicitly. A multitude of methods have been developed in order to obtain approximate solutions, including perturbation, asymptotic and numerical techniques. Even so there remains intense interest in finding exact solutions, for not only are they of intrinsic mathematical value but they can also describe important physical phenomena and, if for no other purpose, they serve as excellent paradigms against which numerical algorithms can be tested and compared.

In this manuscript we study the unsteady Navier–Stokes equations which govern the three-dimensional motion of an incompressible, Newtonian viscous fluid. These equations play a central role in much research within applied mathematics, physics and engineering and, in their most compact form, may be written

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p - \nu\boldsymbol{\nabla}^2\boldsymbol{u} = 0 \qquad (1.1a)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad (1.1b)$$

where $\boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), u_3(\boldsymbol{x},t))$ are the velocity components, $\boldsymbol{x} = (x_1, x_2, x_3)$ denote the usual Cartesian coordinates, $p(\boldsymbol{x},t)$ is the fluid pressure, ν is the kinematic

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viscosity, t the time, $\boldsymbol{\nabla} \equiv (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\boldsymbol{\nabla}^2 \equiv (\partial^2_{x_1}, \partial^2_{x_2}, \partial^2_{x_3})$, with $\partial_{x_1} = \partial/\partial x_1$. Our search for exact solutions of (1.1a, b) will utilize a symmetry reduction technique and, by this, we mean a transformation of the independent and dependent variables such that the number of variables in the transformed system is at least one less than in the original system. Multiple applications of this symmetry reduction technique makes it theoretically possible to reduce the system (1.1a, b) to a system of ordinary differential equations. However, in this work our goal is more modest: the objective here is to reduce the system (1.1a, b), which contains four independent variables, to systems with just three independent variables.

Much of the theory behind symmetry reductions stems from work by Lie who investigated methods of solving ordinary differential equations by turning the problem into one of quadrature. He later extended his ideas to partial differential equations and the salient features as far as we need them are as follows. Suppose we have a second-order system of partial differential equations in m dependent variables $\boldsymbol{u} = (u_1, u_2, \dots, u_m)$ and $n+1$ independent variables $\boldsymbol{x} = (x_0, x_1, x_2, \dots, x_n)$:

$$\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}) = 0 \tag{1.2}$$

where $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ denote the sets of partial derivatives of $\boldsymbol{u}$ of first and second orders, respectively (below we make the identification $x_0 = t$). Then consider the one-parameter continuous Lie group of infinitesimal point transformations of the variables $\boldsymbol{x}$, $\boldsymbol{u}$ given by

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{x} + \varepsilon \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{u}) + \mathcal{O}(\varepsilon^2) \\ \boldsymbol{u}^* &= \boldsymbol{u} + \varepsilon \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{u}) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{1.3}$$

where ε is the group parameter and $\boldsymbol{X} = (X_0, X_1, X_2, \dots, X_n)$ and $\boldsymbol{U} = (U_1, U_2, \dots, U_m)$ are called the *infinitesimals* of the transformation. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\boldsymbol{v} = \sum_{i=0}^{n} X_i \frac{\partial}{\partial x_i} + \sum_{k=1}^{m} U_k \frac{\partial}{\partial u_k}. \tag{1.4}$$

Substitution of (3.2) into (1.2) combined with use of the chain rule and Taylor's theorem shows that the transformed variables $\boldsymbol{x}^*, \boldsymbol{u}^*$ satisfy (1.2) if and only if

$$\mathrm{pr}^{(2)}\boldsymbol{v}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}))|_{\boldsymbol{\Delta}=0} = 0 \tag{1.5}$$

where $\mathrm{pr}^{(2)}\boldsymbol{v}$ is the second prolongation of the vector field (1.4) given by

$$\mathrm{pr}^{(2)}\boldsymbol{v} \equiv \boldsymbol{v} + \sum_{k=1}^{m}\sum_{i=0}^{n} \phi_k^{[i]} \frac{\partial}{\partial u_{k,i}} + \sum_{k=1}^{m}\sum_{i,j=0}^{n} \phi_k^{[ij]} \frac{\partial}{\partial u_{k,ij}}$$

with

$$u_{k,i} \equiv \frac{\partial u_k}{\partial x_i} \qquad u_{k,ij} \equiv \frac{\partial^2 u_k}{\partial x_i \partial x_j} \qquad k = 1, 2, \dots, m, \;\; i, j = 0, 1, \dots, n$$

and $\phi_k^{[i]}$ and $\phi_k^{[ij]}$, respectively, are the infinitesimals associated with $u_{k,i}$ and $u_{k,ij}$, which are explicitly given in terms of $\boldsymbol{X}$ and $\boldsymbol{U}$ and their derivatives. The requirement that the transformation (3.2) maps $\mathcal{S}_{\boldsymbol{\Delta}} := \{\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{\Delta} = 0\}$, the set of solutions of (1.2), into itself, yields an overdetermined system of linear equations for the infinitesimals $\boldsymbol{X}$ and $\boldsymbol{U}$ and the solution of these yields the symmetries of (1.2). The classical method then proceeds to find the associated symmetry reduction by solving the invariant surface conditions

$$\boldsymbol{\psi} \equiv (\boldsymbol{X} \cdot \nabla)\boldsymbol{u} - \boldsymbol{U} = \boldsymbol{0} \tag{1.6}$$

which are a system of quasilinear first-order partial differential equations. The solution of (1.6), when substituted in (1.2), yields a reduced system of equations which depend on at most n independent variables. There are several comprehensive accounts of the classical method (cf [3, 25, 26]).

Bluman and Cole [2] made an important breakthrough in the theory of symmetry reductions and it is their technique that we shall be using here. In their study of symmetry reductions of the linear heat equation, they proposed the so-called 'nonclassical method of group-invariant solutions', which is a generalization of the classical Lie method. To apply the classical method, the condition (1.5) is satisfied by demanding that each of the coefficients of different derivatives are zero, thereby giving rise to a linear system of equations. Bluman and Cole [2] observed that all that is essentially required for a reduction is that

$$\mathrm{pr}^{(2)}(\boldsymbol{\Delta}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}))|_{\boldsymbol{\Delta}=0,\psi=0} = 0. \tag{1.7}$$

Thus one requires that the set of simultaneous solutions of (1.2) and (1.6) is invariant under the transformation (3.2), i.e. the subset of $\mathcal{S}_{\boldsymbol{\Delta}}$ given by $\mathcal{S}_{\boldsymbol{\Delta},\psi} = \{\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{\Delta} = 0, \psi = 0\}$ is invariant under the transformation (3.2). Thus 'nonclassical symmetries' or 'conditional symmetries', of a system of partial differential equations (1.2) are transformations that leave only the subset $\mathcal{S}_{\boldsymbol{\Delta},\psi}$ of the solution set $\mathcal{S}_{\boldsymbol{\Delta}}$ of the system (1.2) invariant; other solutions of (1.2) that are *not* in the subset $\mathcal{S}_{\boldsymbol{\Delta},\psi}$ are *not* necessarily transformed to $\mathcal{S}_{\boldsymbol{\Delta}}$.

This procedure yields an overdetermined, *nonlinear* system of equations, as opposed to a linear system in the classical case, for the infinitesimals $\boldsymbol{X}$ and $\boldsymbol{U}$, which appear in both the transformations (3.2) and the supplementary condition (1.6). The number of determining equations arising in the nonclassical method is smaller than for the classical method, since there are fewer linearly independent expressions in the derivatives. Since all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set may be larger in the nonclassical case. We remark that for some equations the infinitesimals arising from the classical and nonclassical methods coincide. It should be emphasized that the vector fields associated with the nonclassical method do not form a vector space, still less a Lie algebra, since the invariant surface condition (1.6) depends upon the particular reduction. For example, the sum of two nonclassical symmetry operators is not, in general, a symmetry operator at all; similarly the commutator of two nonclassical symmetry operators and the sum of a classical symmetry operator and a nonclassical symmetry operator also are not, in general, symmetry operators.

Both the classical and nonclassical methods have the property that they reduce a system of p equations in m dependent and $n+1$ independent variables to one of at most m dependent and n independent variables; however, the determining equations for the nonclassical method are nonlinear. Furthermore, the nonclassical method has thus far been developed so as to reduce the number of independent variables by precisely one at a time, so that the technique must be used n times in order to reduce a system of partial differential equations in $n + 1$ independent variables to a system of ordinary differential equations. The principal hurdle in the practical implementation of the nonclassical method is the solution of the overdetermined nonlinear system for the infinitesimals. This problem has been somewhat alleviated by developments in the field of differential Gröbner bases and for this work we use ideas based on the work of Mansfield [21] which addresses the issue of overdetermined systems of partial differential equations polynomial in their derivative terms (see also [23, 24]). The calculations described below were executed using the symbolic manipulation package MACSYMA using the program `symmgrp.max` developed by Champagne *et al* [6] which we modified so as to generate the determining equations for the nonclassical method (see also [8, 11]).

Our objective here is to determine systematically all nonclassical reductions of the unsteady, incompressible three-dimensional Navier–Stokes equations (1.1a, b) to systems with three independent variables. We remark that some exact solutions of the Navier–Stokes equations (1.1a, b) which do not arise from classical symmetry reductions have been given by Boisvert *et al* [4], Clarkson [7], Fushchych *et al* [17] and Fushchych and Popowych [15, 16]. However, these investigations are neither complete nor systematic. In fact, Fushchych and Popowych [16] state that an important open problem is the study of nonclassical reductions, which they refer to as conditional symmetries, of the Navier–Stokes equations (1.1a, b). As has already been mentioned we undertake the initial step in the problem by reducing the original system (1.1a, b) with four independent variables to systems with just three independent variables. The nature of the work means that it is virtually impossible to give a detailed description of all the calculations and so, in the main, we restrict ourselves to stating the essential items. Much more detail can be obtained from the first author's PhD thesis [20]. In the next section we give the infinitesimals which arise when one applies the classical Lie method to the Navier–Stokes equations (1.1a, b). In section 3 we consider the nonlinear determining equations for the nonclassical infinitesimals and obtain the associated reductions arising from solving the associated invariant surface conditions (1.6) in section 4; in particular we obtain some new reductions of the Navier–Stokes equations (1.1a, b) which are not obtainable using the classical Lie method. A discussion of the findings and their implications for further work is given in section 5 and we illustrate this with some examples of new exact solutions of the Navier–Stokes equations (1.1a, b) that can be obtained from the reductions given in section 4.

2. Classical symmetries

There have been several investigations into the application of the classical Lie method to the two-dimensional Navier–Stokes equations (cf [5, 18, 27]). However, analogous work for the three-dimensional Navier–Stokes equations is much scarcer. Lloyd [19] and Boisvert *et al* [4] determined the infinitesimals of the Navier–Stokes equations (1.1a, b) as

$$\begin{aligned}\boldsymbol{x} &= c_4\boldsymbol{x} + \boldsymbol{c}\wedge\boldsymbol{x} + \boldsymbol{\tau}(t)\\ T &= 2c_4 t + c_5\\ \boldsymbol{u} &= -c_4\boldsymbol{u} + \boldsymbol{c}\wedge\boldsymbol{u} + \boldsymbol{\tau}'(t)\\ P &= -2c_4 p - \boldsymbol{x}\cdot\boldsymbol{\tau}''(t) + \tau_4(t)\end{aligned} \tag{2.1}$$

where $\boldsymbol{X}$, T, $\boldsymbol{U}$, P are the infinitesimals corresponding to the variables $\boldsymbol{x}$, t, $\boldsymbol{u}$ and p, respectively, $\boldsymbol{c} = (c_1, c_2, c_3)$, $\boldsymbol{\tau}(t) = (\tau_1(t), \tau_2(t), \tau_3(t))$ with c_j, $j = 1, 2, \ldots, 5$, arbitrary constants and τ_k, $k = 1, 2, 3$, arbitrary functions. These results may be used to show that the following five transformations of coordinates leave (1.1a, b) invariant:

$$(\boldsymbol{x}, t, \boldsymbol{u}, p) \longmapsto (\boldsymbol{x}, t+\varepsilon, \boldsymbol{u}, p)a \tag{2.2a}$$

$$(\boldsymbol{x}, t, \boldsymbol{u}, p) \longmapsto (\mathrm{e}^{\varepsilon}\boldsymbol{x}, \mathrm{e}^{2\varepsilon}t, \mathrm{e}^{-\varepsilon}\boldsymbol{u}, \mathrm{e}^{-2\varepsilon}p) \tag{2.2b}$$

$$(\boldsymbol{x}, t, \boldsymbol{u}, p) \longmapsto (\mathbf{O}\boldsymbol{x}, t, \mathbf{O}\boldsymbol{u}, p) \tag{2.2c}$$

$$(\boldsymbol{x}, t, \boldsymbol{u}, p) \longmapsto (\boldsymbol{x}+\boldsymbol{f}(t), t, \boldsymbol{u}+\boldsymbol{f}'(t), p - \boldsymbol{x}\cdot\boldsymbol{f}''(t) - \tfrac{1}{2}\boldsymbol{f}'(t)\cdot\boldsymbol{f}''(t)), \tag{2.2d}$$

$$(\boldsymbol{x}, t, \boldsymbol{u}, p) \longmapsto (\boldsymbol{x}, t, \boldsymbol{u}, p + g(t)) \tag{2.2e}$$

where $\mathbf{O}$ is an arbitrary constant orthogonal matrix and $\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t))$ and $g(t)$ are arbitrary functions. These symmetries represent (2.2a) time symmetries, (2.2b) scalings, (2.2c) rotations, (2.2d), moving coordinates (which include spatial translations and Galilei transformations) and (2.2e) pressure changes. We shall use these invariances to simplify our final forms of reductions and, moreover, it is to be borne in mind that all the reductions quoted subsequently can always be generalized by use of (2.2*a*–*e*).

Fushchych *et al* [17] constructed algorithmically all classical symmetry reductions of (1.1*a*, *b*) which take them directly into ordinary differential equations by calculating inequivalent three-dimensional subalgebras of the Lie algebra of symmetries generated by the transformations (2.2). Subsequently, Fushchych and Popowych [15, 16] determined systematically reductions of (1.1*a*, *b*) to partial differential equations and then extended their results by seeking further reductions of these partial differential equations. After sufficient repetitions of this procedure one is left with only ordinary differential equations. It should be noted that the reduction of partial differential equations of three or more independent variables directly to ordinary differential equations via classical symmetries is in general *not* equivalent to reduction to ordinary differential equations through a sequence of lower-order partial differential equations.

3. The nonclassical infinitesimals

The determination of nonclassical reductions of (1.1*a*, *b*) is equivalent to finding classical reductions of (1.1*a*, *b*) augmented by the conditions

$$(\boldsymbol{X}\cdot\boldsymbol{\nabla})\boldsymbol{u} + T\boldsymbol{u}_t - \boldsymbol{U} = 0 \qquad (\boldsymbol{X}\cdot\boldsymbol{\nabla})p + Tp_t - P = 0 \tag{3.1}$$

where $\boldsymbol{X}$, T, $\boldsymbol{U}$, P, which are all functions of $\boldsymbol{x}$, t, $\boldsymbol{u}$ and p, are the infinitesimals corresponding to the requisite variables (cf [11]). In using the nonclassical method one is at liberty to minimize the necessary computation by appealing to the fact that both $(\boldsymbol{X}, \boldsymbol{U})$ and $(\alpha\boldsymbol{X}, \alpha\boldsymbol{U})$, where α is any function of $\boldsymbol{u}$ and $\boldsymbol{x}$ in the notation of (1.2), define the same invariant surface condition. For a nontrivial condition at least one component of $\boldsymbol{X}$ has to be nonzero and so without loss of generality we take its value to be unity. Bearing this in mind there are essentially two cases to consider depending on whether T vanishes or not. If $T \not\equiv 0$ then we set $T \equiv 1$, without loss of generality, whilst if $T \equiv 0$ then we set $X_3 \equiv 1$, without loss of generality, since there is discrete symmetry in the rotation of subscripts and at least one of X_i, for $i \in \{1, 2, 3\}$, must be nonzero.

3.1. Infinitesimals when $T \equiv 1$

On application of the program `symmgrp.max` to (1.1*a*, *b*) and (1.2) one may back-substitute for u_{1,x_3x_3}, u_{2,x_1x_1} and u_{3,x_2x_2} from (1.1*a*), $u_{1,t}$, $u_{2,t}$, $u_{3,t}$ and p_t from (1.2) and for u_{3,x_3} from (1.1*b*). The program generates 747 determining equations which are then simplified to a comparatively modest 84 equations in seven dependent variables. These 84 equations consist of 71 linear equations (of which 39 are one-term equations and 26 are two-term equations) and 13 nonlinear equations. Solving this reduced set yields two canonical forms of nonlinear infinitesimals (see [20] for further details):

$$\begin{aligned}
\boldsymbol{x} &= \frac{\boldsymbol{x} + \boldsymbol{c}\wedge\boldsymbol{x} + \boldsymbol{\tau}(t)}{2(t-t_0)} \\
T &= 1 \\
\boldsymbol{u} &= \frac{-\boldsymbol{u} + \boldsymbol{c}\wedge\boldsymbol{u} + \boldsymbol{\tau}'(t)}{2(t-t_0)} \\
P &= \frac{-2p - \boldsymbol{x}\cdot\boldsymbol{\tau}''(t) + \tau_4(t)}{2(t-t_0)}
\end{aligned} \tag{3.2a}$$

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{c} \wedge \boldsymbol{x} + \boldsymbol{\tau}(t) \\ T &= 1 \\ \boldsymbol{u} &= \boldsymbol{c} \wedge \boldsymbol{u} + \boldsymbol{\tau}'(t) \\ P &= -\boldsymbol{x} \cdot \boldsymbol{\tau}''(t) + \tau_4(t) \end{aligned} \tag{3.2b}$$

where $\boldsymbol{c} = (c_1, c_2, c_3)$ is an arbitrary constant vector, t_0 is an arbitrary constant and $\boldsymbol{\tau}(t) = (\tau_1(t), \tau_2(t), \tau_3(t))$ is a vector function. These infinitesimals are classical and obtained from (2.1) by setting $c_4 = 1$, $c_5 = -2t_0$ and $c_4 = 0$, $c_5 = 1$, respectively.

3.2. *Infinitesimals when $T \equiv 0$*

In order to ensure that `symmgrp.max` back-substitutes in the most efficient manner, it is expedient to appeal to (3.2a) and thereby recast (1.1b) as

$$U_{1,x_1} + u_{2,x_2} + U_3 - X_1 u_{3,x_1} - X_2 u_{3,x_2} = 0 \tag{3.3a}$$

while also appended to the system (1.1a), (3.2a, b) and (3.3a) are

$$\mathrm{D}_1(u_{1,x_1} + u_{2,x_2} + U_3 - X_1 u_{3,x_1} - X_2 u_{3,x_2}) = 0 \tag{3.3b}$$

$$\mathrm{D}_3(X_1 \boldsymbol{u}_{x_1} + X_2 \boldsymbol{u}_{x_2} + \boldsymbol{u}_3 - \boldsymbol{U}) = 0 \tag{3.3c}$$

$$\mathrm{D}_3(X_1 p_{x_1} + X_2 p_{x_2} + p_{x_3} - P) = 0 \tag{3.3d}$$

where D_i denotes the total differential operator

$$\mathrm{D}_i \equiv \frac{\partial}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i} \frac{\partial}{\partial u_j} + \frac{\partial p}{\partial x_i} \frac{\partial}{\partial p}.$$

The derivatives $u_{1,t}$, $u_{2,t}$ and $u_{3,t}$ are eliminated by back-substitution from (1.1a), u_{1,x_3x_3}, u_{2,x_3x_3}, u_{3,x_3x_3} and p_{x_3} eliminated by use of (3.3c, d), u_{1,x_1x_1} by (3.3b) and u_{1,x_1} from (3.3a). This done, `symmgrp.max` obtains 564 equations which are simplified to 110 equations, which consist of 11 (one-term) linear equations and 99 nonlinear equations. Solving these yields three further canonical infinitesimal forms (see [20] for further details):

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{k}[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)] \\ T &= 0 \\ \boldsymbol{u} &= \boldsymbol{k}[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1'(t)] \\ P &= \tau_1''(t)[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)] + \tau_2(t)[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)]^2 \end{aligned} \tag{3.4a}$$

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{k}[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)] \\ T &= 0 \\ \boldsymbol{u} &= \boldsymbol{k}[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1'(t)] + \frac{\boldsymbol{k}\tau_3(t)}{\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)} \\ P &= \tau_1''(t)[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)] + \tau_2(t)[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)]^2 + \tfrac{1}{2}\tau_3'(t) + \frac{\tau_3^2(t)}{4[\boldsymbol{k} \cdot \boldsymbol{x} + \tau_1(t)]^2} \end{aligned} \tag{3.4b}$$

and

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{c} \wedge \boldsymbol{x} + \boldsymbol{\tau}(t) \\ T &= 0 \\ \boldsymbol{u} &= \boldsymbol{c} \wedge \boldsymbol{u} + \boldsymbol{\tau}''(t) \\ P &= -\boldsymbol{x} \cdot \boldsymbol{\tau}''(t) + \tau_4(t). \end{aligned} \tag{3.4c}$$

We remark that $\boldsymbol{k} \equiv (k_1, k_2, 1)$ where k_1 and k_2 are constants. In addition, for the infinitesimals (3.4b) the vector $\boldsymbol{k}$ must satisfy $\boldsymbol{k} \cdot \boldsymbol{k} = 0$ so that this reduction is complex.

4. Nonclassical reductions

The solution of the invariant surface conditions (1.6) corresponding to the five infinitesimals (3.2a, b), (3.4a–c) given above are best studied in four parts. Rather than set out the details of each calculation, which may be found in [20], for the sake of brevity we restrict ourselves to a list of the final forms of the reductions. One of the difficult issues in these calculations is the decision as to how to select the particular form of each reduction so as to leave the resultant partial differential equations in as concise a form as possible. Bearing in mind that further studies of nonclassical reductions of the Navier–Stokes equations will necessitate consideration of the lower-order partial differential equation systems quoted here, it is desirable to make these as simple as possible. Unfortunately, as with most partial differential equations, the most succinct form of reduction does not necessarily lead to the easiest form of the reduced partial differential equations.

4.1. Reductions arising from infinitesimals (3.2)

We commence with infinitesimals (3.2a).

Reduction 1. Solving the associated surface conditions (1.6) with infinitesimals (3.2a) yields the following reduction

$$\boldsymbol{u}(\boldsymbol{x},t) = t^{-1/2}\mathbf{A}(t)\boldsymbol{v}(\boldsymbol{\xi}) \tag{4.1a}$$

$$p(\boldsymbol{x},t) = t^{-1}q(\boldsymbol{\xi}) \tag{4.1b}$$

where $\boldsymbol{\xi} = t^{-1/2}\mathbf{A}^T(t)\boldsymbol{x}$ and $\boldsymbol{v}(\boldsymbol{\xi})$ and $q(\boldsymbol{\xi})$ satisfy

$$\nabla_\xi \cdot \boldsymbol{v} = 0 \tag{4.1c}$$

$$-\tfrac{1}{2}\mathbf{B}\boldsymbol{v} - \tfrac{1}{2}(\cdot\mathbf{B}\nabla_\xi)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_\xi)\boldsymbol{v} + \nabla_\xi q - \nu\nabla_\xi^2\boldsymbol{v} = \mathbf{0} \tag{4.1d}$$

and the matrices $\mathbf{A}(t)$ and $\mathbf{B}$ are given by either

$$\mathbf{A}(t) = \begin{pmatrix} \cos(\frac{1}{2}c\ln t) & -\sin(\frac{1}{2}c\ln t) & 0 \\ \sin(\frac{1}{2}c\ln t) & \cos(\frac{1}{2}c\ln t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1e}$$

$$\mathbf{B} = \begin{pmatrix} 1 & c & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1f}$$

or

$$\mathbf{A}(t) = \begin{pmatrix} 1-\frac{1}{8}c^2(\ln t)^2 & -\frac{1}{8}\mathrm{i}c^2(\ln t)^2 & \frac{1}{2}c\ln t \\ -\frac{1}{8}\mathrm{i}c^2(\ln t)^2 & 1+\frac{1}{8}c^2(\ln t)^2 & \frac{1}{2}\mathrm{i}c\ln t \\ -\frac{1}{2}c\ln t & -\frac{1}{2}\mathrm{i}c\ln t & 1 \end{pmatrix} \tag{4.1g}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & -\mathrm{i}c \\ c & \mathrm{i}c & 1 \end{pmatrix} \tag{4.1h}$$

with c an arbitrary constant.

Reduction 2. Solving the associated surface conditions (1.6) with infinitesimals (3.2b) yields the reduction

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbf{A}(t)\boldsymbol{v}(\boldsymbol{\xi}) \tag{4.2a}$$

$$p(\boldsymbol{x},t) = q(\boldsymbol{\xi}) \tag{4.2b}$$

where $\boldsymbol{\xi} = \mathbf{A}^T(t)\boldsymbol{x}$ and $\boldsymbol{v}(\boldsymbol{\xi})$, $q(\boldsymbol{\xi})$ satisfy

$$\nabla_\xi \cdot \boldsymbol{v} = 0 \tag{4.2c}$$

$$\boldsymbol{Bv} + (\boldsymbol{\xi} \cdot \mathbf{B}\nabla_\xi)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla_\xi)\boldsymbol{v} + \nabla_\xi q - \nu\nabla_\xi^2 \boldsymbol{v} = \mathbf{0} \tag{4.2d}$$

and $\mathbf{A}(t)$ and $\mathbf{B}$ are matrices given either by

$$\mathbf{A}(t) = \begin{pmatrix} \cos ct & -\sin ct & 0 \\ \sin ct & \cos ct & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.2e}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2f}$$

or

$$\mathbf{A}(t) = \begin{pmatrix} 1 - \frac{1}{2}c^2t^2 & -\frac{1}{2}\mathrm{i}c^2t^2 & ct \\ -\frac{1}{2}\mathrm{i}c^2t^2 & 1 + \frac{1}{2}c^2t^2 & \mathrm{i}ct \\ -ct & -\mathrm{i}ct & 1 \end{pmatrix} \tag{4.2g}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & \mathrm{i}cc \\ -c & -\mathrm{i}c & 0 \end{pmatrix}. \tag{4.2h}$$

4.2. Reductions arising from infinitesimals (3.4a, b)

Reduction 3. Solving the associated surface conditions (1.6) with infinitesimals (3.4*a*) yields the reduction

$$u_1(\boldsymbol{x}, t) = v_1(\boldsymbol{\xi}, t) + k_1 u_3(\boldsymbol{x}, t) \tag{4.3a}$$

$$u_2(\boldsymbol{x}, t) = v_2(\boldsymbol{\xi}, t) + k_2 u_3(\boldsymbol{x}, t) \tag{4.3b}$$

$$u_3(\boldsymbol{x}, t) = [(\boldsymbol{k} \cdot \boldsymbol{x})v_3(\boldsymbol{\xi}, t) - k_1 v_1(\boldsymbol{\xi}, t) - k_2 v_2(\boldsymbol{\xi}, t)]/(\boldsymbol{k} \cdot \boldsymbol{k}) \tag{4.3c}$$

$$p(\boldsymbol{x}, t) = q(\boldsymbol{\xi}, t) + \tau_2(t)(\boldsymbol{k} \cdot \boldsymbol{x})^2/(\boldsymbol{k} \cdot \boldsymbol{k}) \tag{4.3d}$$

where

$$\xi_1 = x_1 - k_1 x_3 \qquad \xi_2 = x_2 - k_2 x_3 \qquad \boldsymbol{k} = (k_1, k_2, 1)$$

with $\boldsymbol{k} \cdot \boldsymbol{k} \neq 0$. Further, $\boldsymbol{v}(\boldsymbol{\xi}, t)$ and $q(\boldsymbol{\xi}, t)$ satisfy the system

$$v_{1,t} + v_1 v_{1,\xi_1} + v_2 v_{1,\xi_2} + (1 + k_1^2)q_{\xi_1} + k_1 k_2 q_{\xi_2} - \nu D^2 v_1 = 0 \tag{4.3e}$$

$$v_{2,t} + v_1 v_{2,\xi_1} + v_2 v_{2,\xi_2} + (1 + k_2^2)q_{\xi_2} + k_1 k_2 q_{\xi_1} - \nu D^2 v_2 = 0 \tag{4.3f}$$

$$2(v_{1,\xi_1}^2 + v_{2,\xi_2}^2 + v_{1,\xi_1} v_{2,\xi_2} + v_{1,\xi_2} v_{2,\xi_1} + \tau_2) + D^2 q = 0 \tag{4.3g}$$

$$v_3 + v_{1,\xi_1} + v_{2,\xi_2} = 0 \tag{4.3h}$$

with

$$D^2 \equiv (1 + k_1^2)\frac{\partial^2}{\partial \xi_1^2} + 2k_1 k_2 \frac{\partial^2}{\partial \xi_1 \partial \xi_2} + (1 + k_2^2)\frac{\partial^2}{\partial \xi_2^2}.$$

Further by (2.2*c*), by rescaling if required, one may choose (k_1, k_2) to be one of the set $\{(0, 0), (\mathrm{i}, \mathrm{i}), (\mathrm{i}, -\mathrm{i}), (-\mathrm{i}, \mathrm{i}), (-\mathrm{i}, -\mathrm{i})\}$. Additionally, it is noted that in the physical situation for which $k_1 = k_2 = 0$, the flow is essentially two-dimensional in x_1, x_2 (and t), supplemented by a linear expansion $u_3 = x_3 v_3(x_1, x_2, t)$.

Reduction 4. Solving the associated surface conditions (1.6) with infinitesimals (3.4*b*) yields the reduction

$$u_1(\boldsymbol{x},t) = k_1 x_3 v_3(\boldsymbol{\xi},t) + v_1(\boldsymbol{\xi},t) \tag{4.4a}$$

$$u_2(\boldsymbol{x},t) = k_2 x_3 v_3(\boldsymbol{\xi},t) + v_2(\boldsymbol{\xi},t) \tag{4.4b}$$

$$u_3(\boldsymbol{x},t) = (x_3 + k_1\xi_1 + k_2\xi_2)v_3(\boldsymbol{\xi},t) - k_1 v_1(\boldsymbol{\xi},t) - k_2 v_2(\boldsymbol{\xi},t) - \frac{\tau_2(t)}{k_1\xi_1 + k_2\xi_2} \tag{4.4c}$$

$$p(\boldsymbol{x},t) = q(\boldsymbol{\xi},t) + \left[\frac{\tau_3'(t)}{2(k_1\xi_1+k_2\xi_2)} + \frac{\tau_3^2(t)}{4(k_1\xi_1+k_2\xi_2)^3} + \tau_2(t)(k_1\xi_1+k_2\xi_2)\right]x_3 \tag{4.4d}$$

where $\xi_1 = x_1 - k_1 x_3$ and $\xi_2 = x_2 - k_2 x_3$, with $k_1^2 + k_2^2 + 1 = 0$. Further, the functions $\boldsymbol{v}(\boldsymbol{\xi},t)$ and $q(\boldsymbol{\xi},t)$ satisfy the system

$$-k_2^2 v_{1,\xi_1} + k_1 k_2 (v_{1,\xi_2} + v_{2,\xi_1}) - k_1^2 v_{2,\xi_2} + 2v_3 - (k_1\xi_1 + k_2\xi_2)(k_1 v_{3,\xi_1} + k_2 v_{3,\xi_2}) + \frac{\tau_3}{(k_1\xi_1+k_2\xi_2)^2} = 0 \tag{4.4e}$$

$$v_{1,t} + v_1 v_{1,\xi_1} + v_2 v_{1,\xi_2} + \left[(k_1\xi_1+k_2\xi_2)v_3 - k_1 v_1 - k_2 v_2 - \frac{\tau_3}{k_1\xi_1+k_2\xi_2}\right] \times (k_1 v_3 - k_1 v_{1,\xi_1} - k_2 v_{1,\xi_2}) + q_{\xi_1} + \nu[D^2 v_1 + 2k_1^2 v_{3,\xi_1} + 2k_1k_2 v_{3,\xi_2}] = 0 \tag{4.4f}$$

$$v_{2,t} + v_1 v_{2,\xi_1} + v_2 v_{2,\xi_2} + \left[(k_1\xi_1+k_2\xi_2)v_3 - k_1 v_1 - k_2 v_2 - \frac{\tau_3}{k_1\xi_1+k_2\xi_2}\right] \times (k_2 v_3 - k_1 v_{2,\xi_1} - k_2 v_{2,\xi_2}) + q_{\xi_2} + \nu[D^2 v_2 + 2k_1k_2 v_{3,\xi_1} + 2k_2^2 v_{3,\xi_2}] = 0 \tag{4.4g}$$

$$v_{3,t} + v_1 v_{3,\xi_1} + v_2 v_{3,\xi_2} + v_3^2 - \left[(k_1\xi_1+k_2\xi_2)v_3 - k_1 v_1 - k_2 v_2 - \frac{\tau_3(t)}{k_1\xi_1+k_2\xi_2}\right] \times (k_1 v_{3,\xi_1} + k_2 v_{3,\xi_2}) - \frac{\tau_3'(t)}{2(k_1\xi_1+k_2\xi_2)^2} - \frac{3\tau_3^2(t)}{4(k_1\xi_1+k_2\xi_2)^4} + \tau_2 + \nu D^2 v_3 = 0 \tag{4.4h}$$

where

$$D^2 \equiv k_2^2 \frac{\partial^2}{\partial\xi_1^2} - 2k_1k_2 \frac{\partial^2}{\partial\xi_1\partial\xi_2} + k_1^2 \frac{\partial^2}{\partial\xi_2^2}.$$

4.3. Reductions arising from infinitesimals (3.4c)

The final possibilities for reductions of the Navier–Stokes equations via the nonclassical method arise through consideration of the fourth set of infinitesimals (3.4*c*). The solution of the invariant surface condition depends on properties of the vectors $\boldsymbol{c}$, $\boldsymbol{k}$ and $\boldsymbol{\tau}(t)$ appearing in the form of the infinitesimals.

Reduction 5. In the generic case when $\boldsymbol{c}\cdot\boldsymbol{\tau}(t) \not\equiv 0$, $\boldsymbol{c} \neq \boldsymbol{0}$ and $c^2 = \boldsymbol{c}\cdot\boldsymbol{c} \neq 0$ we obtain the reduction

$$u_1(\boldsymbol{x},t) = v_1(\boldsymbol{\xi},t)\cos(\tau^{-1}x_3) - v_2(\boldsymbol{\xi},t)\sin(\tau^{-1}x_3) - \tau^{-1}x_2 v_3(\boldsymbol{\xi},t) \tag{4.5a}$$

$$u_2(\boldsymbol{x},t) = v_1(\boldsymbol{\xi},t)\sin(\tau^{-1}x_3) + v_2(\boldsymbol{\xi},t)\cos(\tau^{-1}x_3) + \tau^{-1}x_1 v_3(\boldsymbol{\xi},t) \tag{4.5b}$$

$$u_3(\boldsymbol{x},t) = v_3(\boldsymbol{\xi},t) + \tau'\tau^{-1}x_3 \tag{4.5c}$$

$$p(\boldsymbol{x},t) = q(\boldsymbol{\xi},t) + \tau_4 x_3 - \tfrac{1}{2}\tau''\tau^{-1}x_3^2 \tag{4.5d}$$

where

$$\xi_1 = x_1\cos(\tau^{-1}x_3) + x_2\sin(\tau^{-1}x_3) \qquad \xi_2 = -x_1\sin(\tau^{-1}x_3) + x_2\cos(\tau^{-1}x_3)$$

with $\tau(t) \equiv |\boldsymbol{\tau}(t)|$ and $\tau'(t) \equiv \mathrm{d}\tau/\mathrm{d}t$, and $\boldsymbol{v}(\boldsymbol{\xi}, t)$ and $q(\boldsymbol{\xi}, t)$ satisfy

$$v_{1,\xi_1} + v_{2,\xi_2} + \tau^{-1}\tau' = 0 \tag{4.5e}$$

$$\begin{aligned} &v_{1,t} + v_1 v_{1,\xi_1} + v_2 v_{1,\xi_2} - 2\tau^{-1}v_2 v_3 - \tau^{-2}\xi_1 v_3^2 + 2\tau^{-2}\tau'\xi_2 v_3 + \tau_4\tau^{-1}\xi_2 \\ &\quad + \tau^{-2}[(\tau^2+\xi_2^2)q_{\xi_1} - \xi_1\xi_2 q_{\xi_2}] \\ &\quad - \nu\tau^{-2}[D^2 v_1 - 2(\xi_2 v_{2,\xi_1} - \xi_1 v_{2,\xi_2}) - 2\tau v_{3,\xi_2} - v_1] = 0 \end{aligned} \tag{4.5f}$$

$$\begin{aligned} &v_{2,t} + v_1 v_{2,\xi_1} + v_2 v_{2,\xi_2} + 2\tau^{-1}v_1 v_3 - \tau^{-2}\xi_2 v_3^2 - 2\tau^{-2}\tau'\xi_1 v_3 - \tau_4\tau^{-1}\xi_1 \\ &\quad + \tau^{-2}[(\tau^2+\xi_1^2)q_{\xi_2} - \xi_1\xi_2 q_{\xi_1}] \\ &\quad - \nu\tau^{-2}[D^2 v_2 + 2(\xi_2 v_{1,\xi_1} - \xi_1 v_{1,\xi_2}) + 2\tau v_{3,\xi_1} - v_2] = 0 \end{aligned} \tag{4.5g}$$

$$v_{3,t} + v_1 v_{3,\xi_1} + v_2 v_{3,\xi_2} + \tau^{-1}\tau' v_3 + \tau^{-1}\xi_2 q_{\xi_1} - \tau^{-1}\xi_1 q_{\xi_2} + \tau_4 - \nu\tau^{-2}D^2 v_3 = 0 \tag{4.5h}$$

with

$$D^2 \equiv (\tau^2+\xi_2^2)\frac{\partial^2}{\partial\xi_1^2} - 2\xi_1\xi_2\frac{\partial^2}{\partial\xi_1\partial\xi_2} + (\tau^2+\xi_1^2)\frac{\partial^2}{\partial\xi_2^2} - \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}.$$

Reduction 6. In the case when $\boldsymbol{c}\cdot\boldsymbol{\tau}(t) \not\equiv 0$, $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{c}\cdot\boldsymbol{c} = 0$ we obtain the reduction

$$u_1(\boldsymbol{x}, t) = \tfrac{1}{2}\sqrt{2}\{v_1(\boldsymbol{\xi}, t) + v_2(\boldsymbol{\xi}, t) + s[v_3(\boldsymbol{\xi}, t) + \tau_1' + \tau_2'] + \tfrac{1}{2}s^2[v_2(\boldsymbol{\xi}, t) + \tau_3'] + \tfrac{1}{6}s^3\tau_2'\} \tag{4.6a}$$

$$u_2(\boldsymbol{x}, t) = \tfrac{1}{2}\mathrm{i}\sqrt{2}\{v_1(\boldsymbol{\xi}, t) - v_2(\boldsymbol{\xi}, t) + s[v_3(\boldsymbol{\xi}, t) + \tau_1' - \tau_2'] + \tfrac{1}{2}s^2[v_2(\boldsymbol{\xi}, t) + \tau_3'] + \tfrac{1}{6}s^3\tau_2'\} \tag{4.6b}$$

$$u_3(\boldsymbol{x}, t) = -\mathrm{i}\{v_3(\boldsymbol{\xi}, t) + s[v_2(\boldsymbol{\xi}, t) + \tau_3'] + \tfrac{1}{2}s^2\tau_2'\} \tag{4.6c}$$

$$\begin{aligned} p(\boldsymbol{x}, t) &= q(\boldsymbol{\xi}, t) + s(\tau_4 - \tau_2''\xi_1 - \tau_3''\xi_2) + \tfrac{1}{2}s^2(\tau_2''\xi_2 + \tau_3\tau_3'' - \tau_1\tau_2'' - \tau_2\tau_1'') \\ &\quad + \tfrac{1}{6}s^3(\tau_2\tau_3'' - \tau_3\tau_2'') - \tfrac{1}{24}s^4\tau_2\tau_2'' \end{aligned} \tag{4.6d}$$

where

$$\xi_1 = \tfrac{1}{2}\sqrt{2}(x_1 - \mathrm{i}x_2) - s(\mathrm{i}x_3 + \tau_1) + \tfrac{1}{2}s^2\tau_3 + \tfrac{1}{3}s^3\tau_2 \qquad \xi_2 = -\mathrm{i}x_3 + s\tau_3 + \tfrac{1}{2}s^2\tau_2$$

$$s = \frac{\sqrt{2}\,(x_1 + \mathrm{i}x_2)}{2\tau_2}$$

and $\boldsymbol{v}(\boldsymbol{\xi}, t)$ and $q(\boldsymbol{\xi}, t)$ satisfy the coupled system

$$\tau_2 v_{1,\xi_1} + (\xi_2 - \tau_1)v_{2,\xi_1} + \tau_3 v_{2,\xi_2} - \tau_2 v_{3,\xi_2} + \tau_2' = 0 \tag{4.6e}$$

$$\begin{aligned} &\tau_2 v_{1,t} + v_2(v_3 + \tau_1') + \tau_2 v_1 v_{1,\xi_1} + (\xi_2 - \tau_1)v_2 v_{1,\xi_1} - \tau_2 v_3 v_{1,\xi_2} + \tau_3 v_2 v_{1,\xi_2} \\ &\quad + \tau_4 - \tau_2''\xi_1 - \tau_3''\xi_2 + (\xi_2 - \tau_1)q_{\xi_1} + \tau_3 q_{\xi_2} - \nu[D^2 v_1 + 2v_{3,\xi_1}] = 0 \end{aligned} \tag{4.6f}$$

$$\begin{aligned} &\tau_2 v_{2,t} + \tau_2' v_2 + \tau_2 v_1 v_{2,\xi_1} + (\xi_2 - \tau_1)v_2 v_{2,\xi_1} \\ &\quad + \tau_3 v_2 v_{2,\xi_2} - \tau_2 v_3 v_{2,\xi_2} + \tau_2 q_{\xi_1} - \nu D^2 v_2 = 0 \end{aligned} \tag{4.6g}$$

$$\begin{aligned} &\tau_2 v_{3,t} + v_2(v_2 + \tau_3') + \tau_2 v_1 v_{3,\xi_1} + (\xi_2 - \tau_1)v_2 v_{3,\xi_1} \\ &\quad + \tau_3 v_2 v_{3,\xi_2} - \tau_2 v_3 v_{3,\xi_2} + \tau_2 q_{\xi_2} - \nu[D^2 v_3 + 2v_{2,\xi_1}] = 0 \end{aligned} \tag{4.6h}$$

with

$$D^2 \equiv 2(\xi_2 - \tau_1)\frac{\partial^2}{\partial\xi_1^2} + 2\tau_3\frac{\partial^2}{\partial\xi_1\partial\xi_2} - \tau_2\frac{\partial^2}{\partial\xi_2^2}.$$

Reduction 7. If $\boldsymbol{c}\cdot\boldsymbol{\tau}(t)\equiv 0$ and $\boldsymbol{c}\neq\boldsymbol{0}$ then the pressure field in the reduced form can take one of three identities. Thus we obtain the reduction

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{(\xi_2^2\boldsymbol{c}-\xi_1\boldsymbol{x})v_1(\boldsymbol{\xi},t)+(c^2\boldsymbol{x}-\xi_1\boldsymbol{c})v_2(\boldsymbol{\xi},t)+(\boldsymbol{x}\wedge\boldsymbol{c})v_3(\boldsymbol{\xi},t)}{c^2\xi_2^2-\xi_1^2} \qquad (4.7a)$$

$$p(\boldsymbol{x},t)=q(\boldsymbol{\xi},t)+\begin{cases}\dfrac{\tau_4(t)}{c}\sin^{-1}\left[\dfrac{\xi_1c_k-c^2x_k}{(c_i^2+c_j^2)^{1/2}(c^2\xi_2^2-\xi_1^2)^{1/2}}\right] & \text{if } c\neq 0,\ c_i^2+c_j^2\neq 0\\ \dfrac{c_i\tau_4(t)}{c_jc_k}\ln\left[\dfrac{c_ix_i+c_jx_j}{(c_k^2\xi_2^2-\xi_1^2)^{1/2}}\right] & \text{if } c_i^2+c_j^2=0,\ c_i,c_j,c_k\neq 0\\ \dfrac{\tau_4(t)(c_ix_j-c_jx_i)}{c_i\xi_1} & \text{if } c=0,\ c_k\neq 0\end{cases} \qquad (4.7b)$$

where $(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}$, $\xi_1=\boldsymbol{c}\cdot\boldsymbol{x}$, $\xi_2=(\boldsymbol{x}\cdot\boldsymbol{x})^{1/2}$, $c=(\boldsymbol{c}\cdot\boldsymbol{c})^{1/2}$, and $\boldsymbol{v}(\boldsymbol{\xi},t)$ and $q(\boldsymbol{\xi},t)$ satisfy

$$v_{1,\xi_1}+\xi_2^{-1}v_{2,\xi_2}=0 \qquad (4.7c)$$

$$v_{1,t}+v_1v_{1,\xi_1}+\xi_2^{-1}v_2v_{1,\xi_2}+c^2q_{\xi_1}+\xi_1\xi_2^{-1}q_{\xi_2}-\nu(D^2v_1+2\xi_2^{-1}v_{1,\xi_2})=0 \qquad (4.7d)$$

$$v_{2,t}+v_1v_{2,\xi_1}+\xi_2^{-1}v_2v_{2,\xi_2}+(c^2\xi_2^2-\xi_1^2)^{-1}(-\xi_2^2v_1^2+2\xi_1v_1v_2-c^2v_2^2-v_3^2)$$
$$+\xi_1q_{\xi_1}+\xi_2q_{\xi_2}-\nu(D^2v_2-2v_{1,\xi_1})=0 \qquad (4.7e)$$

$$v_{3,t}+v_1v_{3,\xi_1}+\xi_2^{-1}v_2v_{3,\xi_2}-\tau_4-\nu D^2v_3=0 \qquad (4.7f)$$

with

$$D^2\equiv c^2\frac{\partial^2}{\partial\xi_1^2}+2\xi_1\xi_2^{-1}\frac{\partial^2}{\partial\xi_1\partial\xi_2}+\frac{\partial^2}{\partial\xi_2^2}.$$

Reduction 8. Finally when $\boldsymbol{c}=\boldsymbol{0}$ we obtain the reduction

$$u_1(\boldsymbol{x},t)=(\tau_1'+\tau_1\tau_3'\tau_3^{-1})x_3+v_1(\boldsymbol{\xi},t)+\tau_1v_3(\boldsymbol{\xi},t) \qquad (4.8a)$$

$$u_2(\boldsymbol{x},t)=(\tau_2'+\tau_2\tau_3'\tau_3^{-1})x_3+v_2(\boldsymbol{\xi},t)+\tau_2v_3(\boldsymbol{\xi},t) \qquad (4.8b)$$

$$u_3(\boldsymbol{x},t)=\tau_3'\tau_3^{-1}x_3+v_3(\boldsymbol{\xi},t) \qquad (4.8c)$$

$$p(\boldsymbol{x},t)=q(\boldsymbol{\xi},t)-x_3[\tau_3^{-1}(\tau_1\tau_3)''\xi_1+\tau_3^{-1}(\tau_2\tau_3)''\xi_2-\tau_4]-\tfrac{1}{2}x_3^2[\tau_3^{-1}(\tau_1\tau_3)''\tau_1$$
$$+\tau_3^{-1}(\tau_2\tau_3)''\tau_2+\tau_3^{-1}\tau_3''] \qquad (4.8d)$$

where $\xi_1=x_1-\tau_1x_3$ and $\xi_2=x_2-\tau_2x_3$, and $\boldsymbol{v}(\boldsymbol{\xi},t)$ and $q(\boldsymbol{\xi},t)$ satisfy

$$v_{1,\xi_1}+v_{2,\xi_2}+\tau_3^{-1}\tau_3'=0 \qquad (4.8e)$$

$$v_{1,t}+v_1v_{1,\xi_1}+v_2v_{1,\xi_2}+2\tau_1'v_3+(1+\tau_1^2)q_{\xi_1}$$
$$+\tau_1\tau_2q_{\xi_2}+\tau_1[\tau_3^{-1}(\tau_1\tau_3)''\xi_1+\tau_3^{-1}(\tau_2\tau_3)''\xi_2-\tau_4]-\nu D^2v_1=0 \qquad (4.8f)$$

$$v_{2,t}+v_1v_{2,\xi_1}+v_2v_{2,\xi_2}+2\tau_2'v_3+(1+\tau_2^2)q_{\xi_2}+\tau_1\tau_2q_{\xi_1}$$
$$+\tau_2[\tau_3^{-1}(\tau_1\tau_3)''\xi_1+\tau_3^{-1}(\tau_2\tau_3)''\xi_2-\tau_4]-\nu D^2v_2=0 \qquad (4.8g)$$

$$v_{3,t}+v_1v_{3,\xi_1}+v_2v_{3,\xi_2}+\tau_3^{-1}\tau_3'v_3-\tau_1q_{\xi_1}-\tau_2q_{\xi_2}$$
$$-\tau_3^{-1}(\tau_1\tau_3)''\xi_1-\tau_3^{-1}(\tau_2\tau_3)''\xi_2+\tau_4-\nu D^2v_3=0 \qquad (4.8h)$$

with

$$D^2\equiv(1+\tau_1^2)\frac{\partial^2}{\partial\xi_1^2}+2\tau_1\tau_2\frac{\partial^2}{\partial\xi_1\partial\xi_2}+(1+\tau_2^2)\frac{\partial^2}{\partial\xi_2^2}.$$

5. Discussion

In this work we have initiated the task of seeking nonclassical symmetry reductions of the full Navier–Stokes equations and done so by undertaking a systematic study of forms which take the Navier–Stokes equations to partial differential equations in three independent variables. It is well known that solutions of partial differential equations tend asymptotically to solutions of lower-order equations determined by symmetry reduction [1] and so it is probable that the solutions of our reduced equations will illustrate important physical phenomena. As very few exact solutions of equations (1.1a, b) exist, further information on possible solutions is undoubtedly of significant value.

It is well established that for most partial differential equations the nonclassical method is more general than the classical approach (although, as demonstrated by Bluman and Cole [2] in their study of the one-dimensional heat equation, it is possible that the nonclassical technique will not generate anything additional to the classical results). In order to compare the two methods in the case of the Navier–Stokes equations (1.1a, b), we review quickly the reductions found by Fushchych and Popowych [15, 16] in their comprehensive study of the classical approach. Our reductions 1, 2 and 5 (which is equivalent to reduction 7 when $\boldsymbol{c} \cdot \boldsymbol{c} \neq 0$; $\boldsymbol{c}$ can be taken to be $(0, 0, 1)$ by appeal to symmetry condition (2.2c)) are equivalent to the first three forms listed by Fushchych and Popowych. These authors also obtained most of the results contained within our reductions 5–8. Although it is not at all obvious on initial inspection, if our forms are converted into real canonical coordinates (that is, such that the only second derivatives in the resultant partial differential equations are two-dimensional Laplacian), and if suitable rotations in the two new spatial variables are made, then the forms quoted in [15, 16] are retrieved.

The most dramatic aspect of our work is that reductions 3 and 4 are completely novel and, moreover, cannot be derived by classical means. This is clear when it is noted that the infinitesimals used to find these reductions are not a subset of the invariance conditions (2.2a–e) even allowing for the possibility of multiplication by an arbitrary function of $(\boldsymbol{x}, t, \boldsymbol{u}, p)$. It is to be expected that these reductions could lead to new physically relevant solutions. It is interesting to note that these reductions arise in the case when $T \equiv 0$, where T is the infinitesimal associated with t. We remark that nonclassical reductions which are not obtainable using the classical method in the $T \equiv 0$ case, rather than the generic case when $T \equiv 1$, occur for other multi-dimensional equations (see, for example, [9, 12–14, 22]).

We have emphasized that what has been performed here is the first, but nonetheless vital, step in nonclassical reduction of the Navier–Stokes equations to ordinary differential equations. Although we have been successful in reducing the number of independent variables in our governing system from four to three, the results are too complicated to expect to be able to find direct analytical solutions. Conversely, this reduction means that the numerical solution of the new equations should be much more feasible than a direct simulation of the full forms (1.1a, b). Indeed, even with rapid modern advances in hardware and software, the reliable numerical solution of (1.1a, b) continues to represent an awesome computational task, while study of the two-dimensional Navier–Stokes equations (which has the same number of independent variables as any of the systems listed in section 3) has become much more routine. Numerical investigation of the reduced Navier–Stokes equations would be of great interest.

The next step along the analytical road is clear. Each of the reduced forms given in the preceding section need to be subjected to the nonclassical method in order to seek further systems in two independent variables which in turn could be taken into ordinary differential equations. This is clearly a huge task but in his PhD thesis Ludlow [20] began the process by focusing on equations (4.2*a*–*d*) with **A** and **B** as given by (4.2*e*, *f*). It can be shown that there are five canonical reductions of this system and that four of these are retrievable by classical techniques. The fifth reduction, given by

$$(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), u_3(\boldsymbol{x})) = (v_1(x_1, x_2) - cx_2, v_2(x_1, x_2) + cx_1, x_3 v_3(x_1, x_2)) \tag{5.1a–c}$$

$$p(\boldsymbol{x}) = q(x_1, x_2) + \ell_2 x_3^2 + \tfrac{1}{2}c^2(x_1^2 + x_2^2) \tag{5.1d}$$

where c and ℓ_2 are arbitrary constants, $v_3(x_1, x_2) = -(v_{1,x_1} + v_{2,x_2})$ and $v_1(x_1, x_2)$, $v_2(x_1, x_2)$ and $q(x_1, x_2)$ satisfy

$$-2cv_2 + v_1 v_{1,x_1} + v_2 v_{1,x_2} + q_{x_1} - \nu\boldsymbol{\nabla}^2 v_1 = 0 \tag{5.1e}$$

$$2cv_1 + v_1 v_{2,x_1} + v_2 v_{2,x_2} + q_{x_2} - \nu\boldsymbol{\nabla}^2 v_2 = 0 \tag{5.1f}$$

$$(v_{1,x_1})^2 + v_{1,x_1} v_{2,x_2} + v_2 v_{1,x_1} + (v_{2,x_2})^2 + c(v_{1,x_2} - v_{2,x_1}) + \ell_2 + \tfrac{1}{2}\boldsymbol{\nabla}^2 q = 0 \tag{5.1g}$$

is truly nonclassical. This reduction is new and it is noted that it is invariant under a one-parameter infinitesimal generator of the Navier–Stokes equations; this is a direct consequence of the fact that reduction 2 itself is a classical reduction of (1.1*a*, *b*). Therefore a necessary condition for (2.2) to be obtainable by classical means is that it is invariant under a two- or three-parameter generator of the Navier–Stokes equations. Although this is the case for certain choices of the dependent variables v_1, v_2 and v_3, it is untrue in general.

Ludlow [20] continued his study to examine further reductions of (5.1*a*–*d*) to ordinary differential equations. As well as some well known exact solutions of (1.1*a*, *b*), two particular forms were uncovered.

Example 1. The first new solution of (1.1*a*, *b*) is given by

$$(u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)) = \nu\left(v_1(x_1), x_2 v_2(x_1), -x_3 v_2(x_1) - x_3\frac{\mathrm{d}v_1}{\mathrm{d}x_1}\right) \tag{5.2a}$$

$$p(\boldsymbol{x}, t) = \nu^2\left(-a x_2^2 + \ell_1 x_3^2 + \frac{\mathrm{d}v_1}{\mathrm{d}x_1} - \tfrac{1}{2}v_1^2(x_1)\right) \tag{5.2b}$$

in which a and ℓ_1 are constants and $v_1(x_1)$ and $v_2(x_1)$ satisfy

$$\frac{\mathrm{d}^2 v_2}{\mathrm{d}x_1^2} - v_1\frac{\mathrm{d}v_2}{\mathrm{d}x_1} - v_2^2 + 2a = 0 \tag{5.2c}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1^2}\left(\frac{\mathrm{d}v_1}{\mathrm{d}x_1} + v_2\right) - v_1\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\frac{\mathrm{d}v_1}{\mathrm{d}x_1} + v_2\right) + \left(\frac{\mathrm{d}v_1}{\mathrm{d}x_1} + v_2\right)^2 + 2\ell_1 = 0. \tag{5.2d}$$

Although a number of exact solutions of (5.2*c*, *d*) are known (for example, solutions in which v_1 is linear and v_2 is constant) this generalized form of the reduction is completely new. It is straightforward to scale a from the problem by writing $v_1(x_1) = (2a)^{1/4}V_1(X)$, $v_2(x_1) = (2a)^{1/2}V_2(X)$, $X \equiv (2a)^{1/4}x_1$ and the reduced equations (5.2*c*, *d*) become

$$\frac{\mathrm{d}^2 V_2}{\mathrm{d}X^2} - V_1\frac{\mathrm{d}V_2}{\mathrm{d}X} - V_2^2 + 1 = 0 \tag{5.3a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}X^2}\left(\frac{\mathrm{d}V_1}{\mathrm{d}X} + V_2\right) - V_2\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\mathrm{d}V_1}{\mathrm{d}X} + V_2\right) + \left(\frac{\mathrm{d}V_1}{\mathrm{d}X} + V_2\right)^2 + \frac{\ell_1}{a} = 0. \tag{5.3b}$$

A representative numerical solution of this system is shown in figure 1; the boundary conditions $V_1 = \frac{\mathrm{d}V_1}{\mathrm{d}X} = V_2 = 0$ at $X = 0$ have been chosen as these correspond to zero fluid velocity on the boundary $x_3 = 0$ (see (5.2*a*)). The solutions shown indicate that $|V_1|$ grows linearly as $X \to \infty$ while V_2 asymptotes to a constant in the same limit. Physically, for large arguments the three velocity components are proportional to the three coordinates x_1, x_2 and x_3 so that this solution is indicative of a shearing-type flow structure.

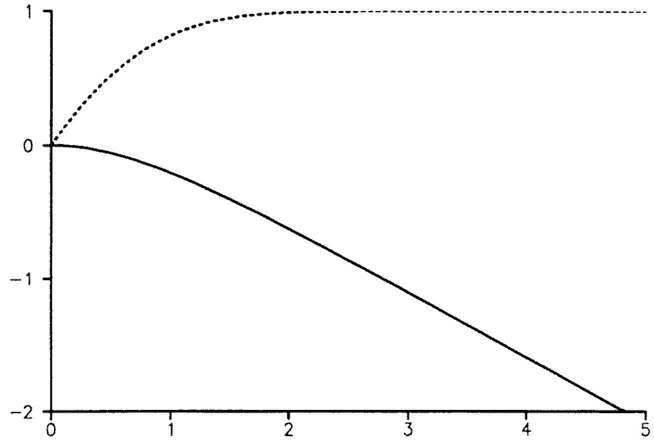

Figure 1. Numerical solution of system (5.3) solved subject to the typical no-slip conditions $V_1 = \frac{dV_1}{dX} = V_2 = 0$ at $X = 0$. The parameter $\ell_1/a = -1$ but the general structure of the solution is insensitive to this ratio. The full curve denotes $V_1(X)/4$; the broken curve $V_2(X)$.

Example 2. The second new solution discovered by Ludlow [20] is given by

$$\begin{pmatrix} u_1(\boldsymbol{x}, t) \\ u_2(\boldsymbol{x}, t) \\ u_3(\boldsymbol{x}, t) \end{pmatrix} = \begin{pmatrix} \cos ct & -\sin ct & 0 \\ \sin ct & \cos ct & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu v_1(\xi_1) - c\xi_2 \\ \nu^2 v_2(\xi_1) - \nu b\xi_2 - c\xi_1 \\ \nu x_3 \left(b - \frac{dv_1}{d\xi_1}\right) \end{pmatrix} \tag{5.4a}$$

$$p(\boldsymbol{x}, t) = -\tfrac{1}{2}(\nu b\xi_2 + 2c\xi_1)^2 + \nu^2 d\xi_2 - \nu^2 \ell_2 x_3^2 + \tfrac{1}{2}c^2(\xi_1^2 + \xi_2^2) - 2\nu^2 q(\xi_1) \tag{5.4b}$$

where b, c, d and ℓ_2 are arbitrary constants, $\xi_1 = x_1 \cos ct + x_2 \sin ct$, $\xi_2 = x_2 \cos ct - x_1 \sin ct$ and $v_1(\xi_1)$, $v_2(\xi_1)$ and $q(\xi_1)$ satisfy

$$\frac{d^3 v_1}{d\xi_1^3} - v_1 \frac{d^2 v_1}{d\xi_1^2} + \left(\frac{dv_1}{d\xi_1} - b\right)^2 - 2\ell_2 = 0 \tag{5.4c}$$

$$\frac{d^2 v_2}{d\xi_1^2} - v_1 \frac{dv_2}{d\xi_1} + bv_2 - d = 0 \tag{5.4d}$$

$$\frac{d^2 v_1}{d\xi_1^2} - v_1 \frac{dv_1}{d\xi_1} + 2cv_2 + 2\frac{dq}{d\xi_1} = 0. \tag{5.4e}$$

The third of these equations is decoupled from the other two and, moreover, some parameters can be removed by re-defining $v_1(\xi_1) = b^{1/2} V_1(X)$, $v_2(\xi_1) = dV_2(X)$ and $X = b^{1/2}\xi_1$. The important equations then become

$$\frac{d^3 V_1}{dX^3} - V_1 \frac{d^2 V_1}{dX^2} + \left(\frac{dV_1}{dX} - 1\right)^2 - \frac{2\ell_2}{b^2} = 0 \tag{5.5a}$$

$$\frac{d^2 V_2}{dX^2} - V_1 \frac{dV_2}{dX} + V_2 - \frac{1}{b^2} = 0. \tag{5.5b}$$

Figure 2 shows numerical solutions of (5.5a, b) with the boundary conditions $V_1 = \frac{dV_1}{dX} = V_2 = 0$ at $X = 0$ imposed and the parameter $\ell_2/b^2 = 2$. As with the previous solution, the component V_1 decreases linearly as $X \to \infty$ while V_2 tends to a constant. We note that the nontrivial structure of the solution of (5.5a, b) is dependent only on the ratio ℓ_2/b^2; the particular value of b^2 in (5.5b) is easily accounted for by scaling of V_2.

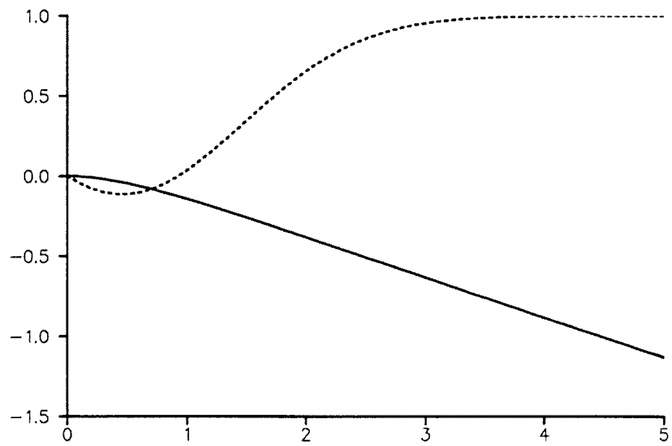

Figure 2. Numerical solution of system (5.5) with $\ell_2 = 2b^2$. The full curve denotes $V_1(X)$; broken curve $V_2(X)$.

This brief account indicates how the form of reduction 2 may be used to generate solutions of the Navier–Stokes equations $(1.1a, b)$ in terms of solutions of ordinary differential equations. For a complete catalogue of nonclassical reductions to ordinary differential equations it is obvious that the same procedure needs to be applied to each of the other reductions listed in section 3. This would inevitably be a very lengthy and substantial task that could only be sensibly envisaged if the most efficient algorithms are implemented. Amongst such considerations is the observation that for further work on reductions 3–8 it would be advantageous to write them in terms of characteristic coordinates and, if this transpires not to be feasible, then a transformation should be applied that minimizes as far as possible the number of occurrences of second-order derivatives. This is essential for efficiency as it leads to a significant reduction in the number of determining equations that arise. Once new reductions are found they need to be subjected to careful analysis which, more than likely, would require the generation of numerical solutions.

To summarize, here we have begun the task of constructing systematically a careful catalogue of nonclassical symmetry reductions of the full unsteady Navier–Stokes equations. Our intention has been to record the full results of the nonclassical reduction of the four-dimensional governing equations to systems involving only three independent variables. Although we have summarized how further work on our reduction 2 leads to novel solutions governed by ordinary differential sets of equations, there is still much to do in relation to the other reductions. To date only a comprehensive study of the classical method as it applies to $(1.1a, b)$ has been accomplished. An open question remains as to whether other symmetry reduction methods may yield yet more information—for example, Clarkson and Kruskal's [10] direct method based on an ansatz approach. (We remark that a review of the more common symmetry reduction techniques and a critique of their various strengths and weaknesses has been compiled by Clarkson [8].) Above all, there undoubtedly remains much information encoded in the symmetry properties of the Navier–Stokes equations yet to be discovered. Further studies of this topic should yield more exact and physically significant solutions of $(1.1a, b)$ which may hold the key to important phenomena in the real world.

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